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Convergence of Fourier Methods for Navier–Stokes Equations

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The Fourier-Galerkin method is used to simulate fluid flows in two and three dimensions, on domains with periodic boundary conditions. It is proved that the numerical solution converges towards the solution of Navier–Stokes equations. The rate of convergence depends on the smoothness of the mathematical solution. Finally, it is shown that the Fourier–Galerkin method can be interpreted as a projection method. This observation may lead to more sophisticated convergence proofs.

INTRODUCTION

The flow of an incompressible fluid satisfies Navier–Stokes equations. Thus, to study turbulent flows—and to test theories about turbulence—one can solve Navier–Stokes equations numerically and instead study the numerical solutions. For convenience the calculations are normally carried out in a square domain with periodic boundary conditions. The solutions can be obtained by either finite difference methods [2, 19, 26], finite element methods [12], vortex methods [6], cloud in cell methods [4] or Fourier methods [11, 13, 14, 29]. The two schemes used most widely to simulate turbulent flows are Arakawa's difference scheme [1] and the Fourier method [30]. In the absence of time-discretization errors and viscous decay both methods conserve energy and enstrophy. However, Kreiss and Oliger [23] have shown that a one-dimensional analogue of Arakawa's difference scheme is unstable if the discretization in time is done by the leapfrog scheme. Since the differential equations describing the flow are non-linear, it is in general difficult to obtain convergence results. Chorin [5] and Temam [32] have proved the convergence for two finite difference schemes. The convergence of some finite element methods is discussed in the book by Temam [34]. Del Prete and Hald [16, 17] have proved the convergence of the vortex method, but only for the inviscid case. The purpose of this paper is to prove the convergence of the Fourier method applied to Navier–Stokes equations with periodic boundary conditions.

The Fourier–Galerkin method can be explained as follows. We expand the solution of Navier–Stokes equations in a Fourier series and write the differential equations as

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an infinite system of ordinary differential equations with the Fourier coefficients as variables. The system is then truncated by setting all Fourier coefficients, except a finite number, identically equal to zero. In this manner Lorentz [27] and Kraichnan [22] have derived small systems to model two-dimensional flow. Computations with large truncations have only become feasible after the development of the fast Fourier transform. Since then the Fourier method has become competitive with finite difference methods and has been used extensively in numerical simulations of turbulence. For studies in two dimensions see [11, 13, 14, 19] and for three dimensions see [29].

There are two versions of the Fourier method; the Fourier–Galerkin method [30] (also called the spectral method) and the collocation method [14] (also called the pseudospectral method). Here our terminology follows Gottlieb and Orszag [15], but it should be mentioned that Kreiss and Oliger [23] introduced the collocation method under the name of the Fourier method. In the collocation method the derivatives of a function are approximated by interpolating the function at a finite number of points by a trigonometric polynomial and evaluating the derivatives of the interpolant exactly. An uncritical use of this technique may lead to unstable schemes; see Kreiss and Oliger [23]. However, in general even the more complicated stable schemes are economically superior to the corresponding Fourier–Galerkin methods, see Fornberg [10] and Kreiss and Oliger [24]. For Navier–Stokes equations with periodic boundary conditions Fox and Orszag [14] have developed a pseudospectral method. Their scheme conserves energy and is therefore stable. There is also a variant of the Fourier–Galerkin technique in which the right-hand side of the differential equations is not calculated exactly by the fast Fourier transform, but aliasing errors are permitted. The amount of calculations can be reduced by a factor two or more, Orszag [30]. The energy for such a scheme may not be conserved.

The convergence of the collocation method to solve first order hyperbolic equations with variable coefficients has been proved by Kreiss and Oliger [23], and the result extended to hyperbolic systems by Fornberg [10]. The convergence of the Fourier–Galerkin method for the same class of problems is straightforward and can be found in Kreiss and Oliger [24] and Gottlieb and Orszag [15]. These convergence results have not been extended to non-linear problems due to the lack of a priori bounds for the solutions of the linearized equations. We circumvent this problem by assuming that the solution of the non-linear differential equation is sufficiently smooth. This approach has been used previously in convergence proofs for Navier–Stokes equations; see Chorin [5] and Hald [17]. In the proofs below we use in addition the fact that the vorticity or the energy is conserved and that the differentiation commutes with the projection in the Galerkin method.

The accuracy of the Fourier method has been investigated empirically on a domain with periodic boundary conditions by Herring *et al.* [19]. They found that with a fixed number of frequencies the Fourier method cannot simulate flows with small viscosity. This conclusion is incompatible with the results presented here. Basically, the viscosity doesn't really matter. The reason is that on a periodic domain the solution of the Navier–Stokes equations tends to the solution of the Euler equations as the viscosity tends to zero; see Ebin and Marsden [7]. It is an open question whether the corresponding result holds for general bounded domains, and the convergence of the Fourier method has not been established for such a case.

In Section 1 we study the convergence of the Fourier method for inviscid, two-dimensional flow. This section is included for pedagogical reasons. The proof is elementary and uses the conservation of vorticity, Gerschgorin's theorem and Minkowsky's inequality. The technique cannot be extended to three dimensions because the vorticity is not conserved in three dimensions. In Section 2 we establish the convergence of the Fourier method for Navier–Stokes equations and for Euler's equations in two and three dimensions. Our basic assumption is that the mathematical solution exists and is smooth. The proof is more abstract than the previous one and based on the conservation of energy.

In Section 3 we will show that the Fourier method is equivalent to a projection method, the so-called Faedo–Galerkin method. This observation is not new (see [8, p. 269]), but it seems that the explicit formula have not been given previously. The projection method can be used to establish the existence of solutions to Navier–Stokes equations. More precisely, it is shown that there is a subsequence of solutions which converges to a weak solution of the differential equations. If this weak solution is a classical solution, then the whole sequence converges. Thus if the initial data are sufficiently smooth, then the Fourier method converges for the Navier–Stokes equations for all time in two dimensions, and in three dimensions it converges for a finite time, which depends on the initial data, but is independent of ν. To prove these propositions it is necessary to check that all the arguments for Navier–Stokes equations for a compact domain with smooth boundary can be carried over (or modified) to a domain with periodic boundary conditions. This can be done, but we shall not present any details; see Boldrighini [3, Chap. 9]. The existence of solutions of Euler's equations for bounded domains in R^3 has been considered by Temam [33] and Foias *et al.* [9]. The convergence of the Fourier method for Euler's equations does not follow simply by letting the viscosity tend to zero. The reason is that the a priori bounds for the solutions of Navier–Stokes equations depend on ν^{-1} and explode as ν tends to zero. However, if the initial data are sufficiently smooth, then the viscous flow tends toward the inviscid flow as the viscosity tends to zero. This provides an indirect proof for the convergence of the Fourier method for Euler's equations, but gives no indication of the rate of convergence.

1. Convergence in Two Dimensions

In this section we will prove the convergence of the Fourier method for Euler's equations in two dimensions. The convergence proof for the Navier–Stokes equations is almost identical. We consider the flow of a two-dimensional incompressible, inviscid fluid in a square domain with side length 2π and with periodic boundary conditions. By expanding the vorticity $\zeta(x, y, t)$ in a Fourier series we find from Euler's equations that the Fourier coefficients $\zeta_k(t)$ of ζ satisfy

$$\dot{\zeta}_k = \sum_{p+q=k} C_{pq}\zeta_p\zeta_q \tag{1.1}$$

for $k=(k_1, k_2)\neq 0$ and that $\zeta_0 \equiv 0$. Here $C_{pq}=\frac{1}{2}(|q|^{-2}-|p|^{-2})\,|p,q|$, where $p=(p_1, p_2)$, $|p|^2=p_1^2+p_2^2$ and $|p,q|=p_1q_2-p_2q_1$. The sum is taken over all integer values of p_1 and p_2 with $p\neq 0$. Since the vorticity is real we have $\zeta_{-k}=\bar{\zeta}_k$ for all k. If the initial vorticity is three times continuously differentiable, then the solution of Eq. (1.1) is unique and exists for all time, see Ebin and Marsden [7].

Let F be the set of frequencies k for which $|k|\leqslant N$. To truncate Eq. (1.1) we set $\zeta_k\equiv 0$ for all k not in F and for all time. Thus we arrive at the Fourier–Galerkin method

$$\dot{\omega}_k = \sum_{\substack{p+q=k\\ p,q\in F}} C_{pq}\omega_p\omega_q \tag{1.2}$$

for all k in F. If $\omega_{-k}=\bar{\omega}_k$ at $t=0$ then it will be satisfied for all time t. We can now formulate

THEOREM 1. *Let ζ_k and ω_k be the solutions of* (1.1) *and* (1.2). *Let* $2C=\max_{0\leqslant\tau\leqslant t}\sum|k|\,|\zeta_k(\tau)|$. *If $\omega_k=\zeta_k$ for all k in F at $t=0$ and $\zeta_0=0$ then*

$$\left(\sum_{k\in F}|\omega_k(t)-\zeta_k(t)|^2\right)^{1/2}\leqslant\frac{2e^{Ct}}{N^2}\max_{0\leqslant\tau\leqslant t}\left(\sum_{k\notin F}|k|^6\,|\zeta_k(\tau)|^2\right)^{1/2}.$$

Remark. If $\zeta(x, y, 0)$ is three times continuously differentiable then C and the sum $\sum|k|^6|\zeta_k|^2$ are bounded for all finite t. The Fourier method will therefore converge as N tends to infinity.

Proof. Let $e_k=\omega_k-\zeta_k$ be the error in the kth Fourier coefficient. It follows from Eqs. (1.1) and (1.2) that

$$\dot{e}_k=\sum_{\substack{p+q=k\\ p,q\in F}} C_{pq}(e_pe_q+\zeta_qe_p+\zeta_pe_q)-r_k,$$

$$r_k=\left(\sum_{\substack{p+q=k\\ p\in F,q\notin F}}+\sum_{\substack{p+q=k\\ p\notin F,q\in F}}+\sum_{\substack{p+q=k\\ p\notin F,q\notin F}}\right)C_{pq}\zeta_p\zeta_q.$$

To measure the error we set $\varepsilon^2=\sum_{k\in F}\bar{e}_ke_k$. By differentiating ε^2 wrt t we see that

$$(\varepsilon^2)^{\cdot}=\sum_{k\in F}\bar{e}_k\sum_{\substack{p+q=k\\ p,q\in F}} C_{pq}(e_pe_q+2\zeta_qe_p)-\sum_{k\in F}\bar{e}_kr_k+\text{c.c.},$$

where c.c. means the complex conjugate expression and we have used that $C_{pq} = C_{qp}$. The conservation of vorticity implies that $\sum C_{pq}\bar{e}_k e_p e_q + \text{c.c.}$ is zero. Let $e = (e_k)$ and $r = (r_k)$. We can then rewrite the expression for the error in matrix notation

$$(\varepsilon^2)^{\cdot} = 2e^*(A + A^*)e - e^*r - r^*e, \tag{1.3}$$

where e^* is the complex conjugate transpose of e and the kpth element of $A + A^*$ is $C_{kp}\zeta_{k-p}$ if $k - p$ is in F and zero otherwise. To estimate the 2-norm of $A + A^*$ we use Gerschgorin's theorem. We observe first that $|k, p| = |k - p, p| \leqslant |k - p|\,|p|$ for $|p| < |k|$ and that $|k, p| \leqslant |k|\,|p - k|$ for $|p| > |k|$. We can therefore bound the absolute row sum of the kth row of $A + A^*$ by

$$\sum_{p,k-p\in F} \frac{1}{2}\left|\left(\frac{1}{|p|^2} - \frac{1}{|k|^2}\right)|k, p|\,\zeta_{k-p}\right| \leqslant \frac{1}{2}\sum |k - p|\,|\zeta_{k-p}| \leqslant C.$$

Thus $\|A + A^*\|_2 \leqslant C$ as $A + A^*$ is Hermitian and we conclude from Eq. (1.3) that

$$2\varepsilon\dot{\varepsilon} \leqslant 2\varepsilon^2 C + 2\varepsilon\,\|r\|_2,$$

where $\|r\|_2^2 = \sum_{k\in F} |r_k|^2$. By solving this differential inequality we get

$$\varepsilon(t) \leqslant \frac{e^{Ct} - 1}{C} \max_{0\leqslant\tau\leqslant t} \|r(\tau)\|_2. \tag{1.4}$$

Our next step is to estimate the truncation error r. Since $2\,|C_{pq}| \leqslant |p|\,|q|$ we see that

$$\begin{aligned} |r_k| &\leqslant \left|\sum_{\substack{p+q=k\\ p\in F, q\notin F}} C_{pq} 2\zeta_p\zeta_q\right| + \left|\sum_{\substack{p+q=k\\ p\notin F, q\notin F}} C_{pq}\zeta_p\zeta_q\right| \\ &\leqslant \sum_{\substack{p+q=k\\ q\notin F}} |p|\,|\zeta_p|\,|q|\,|\zeta_q|. \end{aligned}$$

To estimate $\|r\|_2$ we use Minkowsky's inequality; see [18, p. 123]. Let $\xi_q = \zeta_q$ if q is not in F and zero otherwise. Then

$$\begin{aligned} \|r\|_2 &\leqslant \left(\sum_k \left(\sum_p |p|\,|\zeta_p|\,|k - p|\,|\xi_{k-p}|\right)^2\right)^{1/2} \\ &\leqslant \sum_p |p|\,|\zeta_p| \left(\sum_k |k - p|^2\,|\xi_{k-p}|^2\right)^{1/2} \\ &\leqslant 2C\left(\sum_{q\notin F} |q|^2\,|\zeta_q|^2\right)^{1/2}. \end{aligned}$$

By combining this result with (1.4) and using $|q|^2 < |q|^6/N^4$ for all q not in F we arrive at the statement in the theorem. This completes the proof.

The technique presented in this section does not extend to three dimensions because the vorticity is not conserved in three dimensions. However, a proof using instead the conservation of energy can be generalized. In Section 2 we will present such a proof, but in a more abstract language. Theorem 1 implies the convergence of the computed velocity field, with rate of convergence $1/N^2$. This result is not optimal and is improved in the next section.

2. Convergence in Two and Three Dimensions

In this section we will prove the convergence of the Fourier method for Euler's equations and for Navier–Stokes equations in two and three dimensions. We formulate the result for Navier–Stokes equations in three dimensions, but most of the proof is independent of the dimension. The flow of an incompressible, viscous fluid satisfy the Navier–Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \tag{2.1}$$

$$\operatorname{div} u = 0, \tag{2.2}$$

where $u = u(x, t)$ is the velocity, p is the pressure and ν is the viscosity. We use the notation $u \cdot \nabla = \sum u_j D_j$ and $\Delta u = \sum D_j^2 u$ with $D_j u = \partial u / \partial x_j$. We look for solutions $u = (u_i)$ and p with period 2π in each space variable. By taking the divergence of (2.1) and using Eq. (2.2) we see that

$$\Delta p = -\nabla \cdot (u \cdot \nabla)u. \tag{2.3}$$

The pressure is therefore determined up to a constant and can be eliminated. We say that the solution u is in $H^m(T_n)$ if $D^\alpha u$ is square integrable over T_n for all $|\alpha| = m$. Here $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and T_n is a cube in R^n with sidelength 2π. Ebin and Marsden [7] have shown that if the initial data $u(x, 0)$ are in $H^m(T_n)$ with $m > n/2 + 5$ and satisfy Eq. (2.2), then there exists a unique solution of (2.1–2.2) for a short time interval (independent of ν). The solution $u(x, t)$ is in H^m and differentiable wrt t. Moreover, the solution of Navier–Stokes equations converges to the solution of Euler's equations as the viscosity tends to zero. Similar results have been obtained for R^2 and R^3 by McGrath [28] and Kato [21].

The Fourier method can be derived as follows. We expand the solution u in a Fourier series and let P be the projection

$$Pu = P \sum u_k e^{ik \cdot x} = \sum_{k \in F} u_k e^{ik \cdot x},$$

where F is the set of frequencies k for which $|k| < N$. Let $Q = I - P$. If $v = Pv$ for all t then the Fourier–Galerkin method is

$$\frac{\partial v}{\partial t} + P[(v \cdot \nabla)v] = -\nabla q + \nu \Delta v, \tag{2.4}$$

$$\operatorname{div} v = 0. \tag{2.5}$$

Our convergence proof depends on three properties of P, namely, that $Pu \to u$ as $N \to \infty$, that $D_j P = P D_j$ and that $\int v \cdot Pu = \int v \cdot u$. Since P commutes with differentiation we see that the pressure q can be eliminated by the analogue of Eq. (2.3)

$$\Delta q = -P[\nabla \cdot (v \cdot \nabla)v]. \tag{2.6}$$

It follows from Eqs. (2.4) and (2.6) that the Fourier coefficients v_k of v satisfy a system of ordinary differential equations, namely,

$$\left(\frac{d}{dt} + \nu |k|^2\right) v_k = -i \sum_{\substack{p+q=F \\ p,q \in F}} k^T v_p (I - kk^T/|k|^2)\, v_q; \tag{2.7}$$

see Kraichnan [22]. The initial conditions are given by $v = Pu$ and satisfy Eq. (2.5). In addition we assume that $u_0 = 0$. This assumption is both natural and convenient. It implies that $u_0 = v_0 \equiv 0$ for all time, and this fact simplifies our proof a little. On a deeper level the assumption makes the theory of Navier–Stokes equation on domains with periodic boundary conditions similar to the theory for compact domains. For example, in the periodic case the viscous solution tends to u_0 as t tends to infinity, whereas the solution for bounded domains always tend to zero, see [34, p. 318]. More important is that if $u_0 \neq 0$ then Poincaré's inequality fails and this inequality is used repeatedly in the mathematical theory.

THEOREM 2. *Let u and v be the solutions of (2.1–2.2) and (2.4–2.5) with $u_0 = v_0 = 0$ and $n = 2$ or 3. Let $A = \max_{0 \leqslant \tau \leqslant t} \sum |k|\,|u_k|$. If $u(x, 0)$ is in $H^m(T_n)$ with $m \geqslant 3$ then*

$$\|v(t) - u(t)\|_2 \leqslant \frac{2e^{At} n^{m/2}}{N^{m-1}} \max_{\substack{0 \leqslant \tau \leqslant t \\ |\alpha| = m}} \|D^\alpha Q u\|_2.$$

Let $\zeta = \nabla \times u$ and $\omega = \nabla \times v$. If $\zeta(x, 0)$ is in $H^m(T_n)$ with $m \geqslant 2$, then $A = \max_{0 \leqslant \tau \leqslant t} \sum |\zeta_k|$ and

$$\|\omega(t) - \zeta(t)\|_2 \leqslant \frac{2e^{At} n^{m/2}}{N^{m-1}} \max_{\substack{0 \leqslant \tau \leqslant t \\ |\alpha| = m}} \|D^\alpha Q \zeta\|_2.$$

Remark. The assumption on u implies that $A < \infty$, [31, p. 249]. Thus the Fourier method converges as N tends to ∞, provided the mathematical solution exists and is sufficiently smooth. Note that ζ is in H^m iff u is in H^{m+1}. The rate of convergence of the vorticity is the same as in Theorem 1, but the bound is sharper, since in general $A < C$.

Proof. Let $w = v - Pu$ be the error. By applying P to both sides of Eq. (2.1) and using Eq. (2.4) we see that

$$\frac{\partial w}{\partial t} + P[w \cdot \nabla w + Pu \cdot \nabla w + w \cdot \nabla Pu] = -\nabla(q - Pp) + \nu \Delta w + Pr, \qquad (2.8)$$

where $r = u \cdot \nabla u - Pu \cdot \nabla Pu$. To estimate the error we introduce two inner products and the corresponding norms

$$(u, v) = \sum_{i=1}^{n} \int u_i \bar{v}_i, \qquad \|u\|_2 = \left(\sum_i \|u_i\|^2\right)^{1/2}$$

$$((u, v)) = \sum_{i,j=1}^{n} \int D_j u_i D_j \bar{v}_i, \qquad \|u\|_H = \left(\sum_{i,j} \|D_j u_i\|^2\right)^{1/2}.$$

Note that $\|\cdot\|_H$ is really a norm because $u_0 = v_0 = 0$. By using Eq. (2.8) we see that

$$\frac{1}{2}\frac{d}{dt}(w, w) = (w, -P[w \cdot \nabla w + Pu \cdot \nabla w + w \cdot \nabla Pu] - \nabla(q - Pp) + \nu \Delta w + Pr).$$

Since $(w, Pu) = (w, u)$ and $\operatorname{div} w = 0$ we find after integrating by parts that the contributions from the first, second and fourth term on the right hand side of this equation are zero. Thus

$$\frac{1}{2}\frac{d}{dt}(w, w) = -(w, w \cdot \nabla Pu) - \nu((w, w)) + (w, r).$$

From this point on, the proof is similar to a convergence proof for a linear problem. By using Schwarz inequality and $\nu \geqslant 0$ we find that

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 \leqslant \sum_{i,j} \|w_i\| \, \|w_j\| \, \|D_j P u_i\|_\infty + \sum_i \|w_i\| \, \|r_i\|. \qquad (2.9)$$

To complete the proof we use that the Fourier series for $D_j u$ converges. Since $\|D_j P u_i\|_\infty \leqslant \sum |k_j| \, |u_{ik}|$ we can estimate the first term in (2.9) by

$$\sum_k \sum_i \|w_i\| \, |u_{ik}| \sum_j \|w_j\| \, |k_j| \leqslant \sum_k |k| \, |u_k| \, \|w\|_2^2 \leqslant A \, \|w\|_2^2. \qquad (2.10)$$

Our next goal is to estimate the truncation error. Since $r = Qu \cdot \nabla u - Pu \cdot \nabla Qu$ where $Q = I - P$ we see that the last term in (2.9) is less than

$$\sum_{i,j} \|w_i\| \, \|Qu_j\| \, \|D_j u_i\|_\infty + \sum_{i,j} \|w_i\| \, \|Pu_j\|_\infty \, \|D_j Qu_i\|. \tag{2.11}$$

The first term is treated like (2.10) and less than $A \, \|Qu\|_2 \, \|w\|_2$. The second term in (2.11) can be estimated as

$$\sum_{j,k} |u_{jk}| \sum_i \|w_i\| \, \|D_j Qu_i\| \leqslant \sum_k \sum_j |u_{jk}| \, \|D_j Qu\|_2 \, \|w\|_2$$
$$\leqslant \sum_k |u_k| \, \|Qu\|_H \, \|w\|_2.$$

Since $\sum_k |u_k| \leqslant A$ and $\|Qu\|_2 \leqslant \|Qu\|_H$ we see that (2.11) is less than $2A \, \|Qu\|_H \, \|w\|_2$. Thus it follows from (2.9) and (2.10) that

$$\|w\|_2 \frac{d}{dt} \|w\|_2 \leqslant A \, \|w\|_2^2 + 2A \, \|Qu\|_H \, \|w\|_2.$$

By solving this differential inequality we get

$$\|w\|_2 \leqslant 2(e^{At} - 1) \max_{0 \leqslant \tau \leqslant t} \|Qu\|_H. \tag{2.12}$$

To estimate the rate of convergence we use the smoothness of u. It follows from Parseval's theorem that

$$\|Qu\|_H^2 = (2\pi)^n \sum_{k \notin F} |k|^2 \, |u_k|^2$$
$$\leqslant \frac{(2\pi)^n}{N^{2m-2}} \sum_{k \notin F} (k_1^2 + \cdots + k_n^2)^m \, |u_k|^2$$
$$= \frac{1}{N^{2m-2}} \sum_{|\alpha| = m} \frac{m!}{\alpha!} (2\pi)^n \sum_{k \notin F} k^{2\alpha} \, |u_k|^2$$
$$\leqslant \frac{n^m}{N^{2m-2}} \max_{|\alpha| = m} \|D^\alpha Qu\|_2^2.$$

Here $k^{2\alpha} = k_1^{2\alpha_1} \cdots k_n^{2\alpha_n}$. We can now estimate $v - u$. Since $\|Qu\|_2 \leqslant \|Qu\|_H$ we see that

$$\|v - u\|_2 \leqslant \|v - Pu\|_2 + \|Pu - u\|_2$$
$$\leqslant 2(e^{At} - 1) \max_{0 \leqslant \tau \leqslant t} \|Qu\|_H + \|Qu\|_2$$
$$\leqslant 2e^{At} \frac{n^{m/2}}{N^{m-1}} \max_{\substack{0 \leqslant \tau \leqslant t \\ |\alpha| = m}} \|D^\alpha Qu\|_2.$$

This completes the proof for the convergence of the velocity field. We will now study the convergence of the vorticity field. Since $\zeta = \nabla \times u$ and $\operatorname{div} u = 0$ we see that the Fourier coefficients ζ_k of ζ satisfy $|\zeta_k| = |k|\,|u_k|$. Thus it follows from Parseval's theorem that $\|\zeta\|_2 = \|u\|_H$. By using that $|k| \leqslant N$ for k in F, Eq. (2.12), Parseval's theorem and that $|k| > N$ for k not in F we get

$$\begin{aligned} \|\omega - P\zeta\|_2 &= \|v - Pu\|_H \leqslant N\,\|w\|_2 \\ &\leqslant N \cdot 2(e^{At} - 1) \max \|Qu\|_H \\ &= 2(e^{At} - 1) \max N\,\|Q\zeta\|_2 \\ &\leqslant 2(e^{At} - 1) \max_{0 \leqslant \tau \leqslant t} \|Q\zeta\|_H. \end{aligned}$$

This inequality corresponds to (2.12) and the final result is obtained by repeating the previous arguments with u replaced by ζ. This completes the proof.

The convergence of the velocity field is valid for higher space dimensions provided $m > n/2 + 1$. By copying the mathematical theory for the Navier–Stokes equations we can prove the convergence under weaker smoothness assumptions on the data; see [25] and [34]. There is also a rich opportunity to use sophisticated results for the convergence of Fourier series. It is reasonable that the numerical solution is less accurate than simply truncating the Fourier series of the solution. The loss of one derivative also occurs in the error estimates for the solution of hyperbolic systems obtained by the Fourier method, see [24]. Our proof is based on the cutoff $|k| \leqslant N$. If another cutoff is used then N should be interpreted as the smallest value of $|k|$ for k not in F. In addition, the error bound for the vorticity will have an extra factor, namely, $\max_{k \in F} |k| / \min_{k \notin F} |k|$. If the cutoff is $|k_i| < N$ for $i = 1, \ldots, n$, then this factor is $n^{1/2}$. For the cutoff suggested by Orszag [30], the octodecahedron, the factor is $2/3^{1/2}$.

3. Reformulation of the Fourier Method

In this section we will show that the results of the Fourier method can be interpreted as obtained by the projection method. It is not obvious that the two methods are equivalent. In the projection method the approximate solution is written as a linear combination of divergence free linearly independent functions. In the Fourier method the pressure is determined such that the divergence of the solution is zero, but the basis functions are linearly dependent. We will give a set of orthonormal basis functions and show that the approximate solution of the projection method satisfies the differential equations for the Fourier method. We will present the arguments in three dimensions, and mention the results in two dimensions.

Since $\operatorname{div} v = 0$ and $u_0 = 0$ there exists a function B such that $\operatorname{div} B = 0$ and

$u = \nabla \times B$. In addition $\Delta B = -\zeta$, where $\zeta = \nabla \times u$. By expanding B in a Fourier series we find

$$u = \nabla \times B = \sum ik \times b_k e^{ik\cdot x} = \sum_{k \neq 0} iS_k b_k e^{ik\cdot x}$$

and $u_k = iS_k b_k$. It is our goal to express $S_k b_k$ as $|k| A_k \beta_k$, where β_k is a vector with two components, $k \cdot A_k = 0$ and the columns of A_k are orthonormal. Let $\alpha = k/|k|$. Then

$$S_k = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} = |k| \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix} = |k|\, S_\alpha.$$

Observe that $S_\alpha^T = -S_\alpha$ and that $S_\alpha \alpha = 0$. To find A_k we use the singular value decomposition of S_α, i.e., $S_\alpha = U\Lambda V^T$, where U and V are orthonormal and Λ is a real diagonal matrix. Since V consists of the eigenvectors of $S^T S$ and $S^T S = I - \alpha\alpha^T$ we see that α is the eigenvector corresponding to the eigenvalue 0 and the other two eigenvectors correspond to the eigenvalue 1. To determine V we use Householder's transformation, i.e., $V = I - 2ww^T$, where $w^T w = 1$. Let $(I - 2ww^T)\, e_3 = -\sigma\alpha$, where $e_{3i} = \delta_{i3}$. Here $\sigma = \operatorname{sign}(\alpha_3)$ if $\alpha_3 \neq 0$. If $\alpha_3 = 0$ then $\sigma = \operatorname{sign}(\alpha_2)$. If $\alpha_2 = \alpha_3 = 0$ then we set $\sigma = \operatorname{sign}(\alpha_1)$. Thus

$$V = \frac{1}{1+|\alpha_3|} \begin{bmatrix} \alpha_2^2 + \alpha_3^2 + |\alpha_3| & -\alpha_1\alpha_2 & -\sigma\alpha_1(1+|\alpha_3|) \\ -\alpha_1\alpha_2 & \alpha_1^2 + \alpha_3^2 + |\alpha_3| & -\sigma\alpha_2(1+|\alpha_3|) \\ -\sigma\alpha_1(1+|\alpha_3|) & -\sigma\alpha_2(1+|\alpha_3|) & -\sigma\alpha_3(1+|\alpha_3|) \end{bmatrix}.$$

The complicated definition of σ implies that V does not change if α is replaced by $-\alpha$. To determine U we use that $SV = U \operatorname{diag}(1, 1, 0)$. The last column of U is undetermined, but we may take

$$U = \frac{1}{1+|\alpha_3|} \begin{bmatrix} -\sigma\alpha_1\alpha_2 & -\sigma(\alpha_2^2 + \alpha_3^2 + |\alpha_3|) & \alpha_1(1+|\alpha_3|) \\ \sigma(\alpha_1^2 + \alpha_2^2 + |\alpha_3|) & \sigma\alpha_1\alpha_2 & \alpha_2(1+|\alpha_3|) \\ -\alpha_2(1+|\alpha_3|) & \alpha_1(1+|\alpha_3|) & \alpha_3(1+|\alpha_3|) \end{bmatrix}.$$

This completes the singular value decomposition of S_α. Let $V = (v_1, v_2, v_3)$. Since $V = V^T$ we see that $S_k b_k = |k| A_k \beta_k$, where $A_k = (a_{ij}^{(k)})$ are the first two columns of U and $\beta_{1k} = v_1 \cdot b_k$ and $\beta_{2k} = v_2 \cdot b_k$. Note that $v_3 \cdot b_k = 0$ as $\operatorname{div} B = 0$. We can now write

$$u = \sum_k i\,|k|\, A_k \beta_k(t)\, e^{ik\cdot x}. \tag{3.1}$$

Since the matrix U is orthonormal we have found a representation of u in which the basis functions are orthogonal and divergence free. The corresponding representation in two dimensions is

$$u = \sum_{k \neq 0} \frac{i}{|k|^2} \begin{pmatrix} k_2 \\ -k_1 \end{pmatrix} \zeta_k(t)\, e^{ik \cdot x},$$

where ζ_k are the Fourier coefficients of the vorticity $\zeta = D_1 u_2 - D_2 u_1$.

The projection method is based on the weak formulation of Navier–Stokes equations. Assume that u and p are the solution of the Navier–Stokes equations and let v be a smooth, 2π periodic function which satisfies $\operatorname{div} v = 0$. By multiplying Eq. (2.1) with $\bar{v}$, integrate over T_3, and integrate by parts on the right-hand side of the equation we see that

$$\frac{d}{dt} \sum_i \int u_i \bar{v}_i + \nu \sum_{i,j} \int D_j u_i D_j \bar{v}_i + \sum_{i,j} \int u_j D_j u_i \bar{v}_i = 0. \tag{3.2}$$

Note that the pressure term has disappeared. We look for an approximate solution u of the form (3.1), where the sum is restricted to all k in F. To determine the coefficients $\beta_k(t)$ we require that Eq. (3.2) be satisfied for all test functions v with components $v_i = a_{ij}^{(k)} \exp(ik \cdot x)$, where k is in F and $j = 1$ or 2. This construction of an approximate solution of Navier–Stokes equations goes back to Hopf [20]. By using the orthogonality of the test functions we get

$$\left(\frac{d}{dt} + \nu |k|^2\right) \beta_k = |k|^{-1} \sum_{\substack{p+q=k \\ p,q \in F}} |p|\,|q|\, k^T A_p \beta_p A_k^T A_q \beta_q. \tag{3.3}$$

At first sight this non-linear system of ordinary differential equations seems quite different from Eq. (2.7). However, by multiplying both sides of Eq. (3.3) with $i\,|k|\, A_k$ and using that $u_k = i\,|k|\, A_k \beta_k$ we obtain

$$\left(\frac{d}{dt} + \nu |k|^2\right) u_k = -i \sum_{\substack{p+q=k \\ p,q \in F}} k^T u_p A_k A_k^T u_q.$$

Since $U_\alpha = (A_k \mid \alpha)$ and $U_\alpha U_\alpha^T = I$ we see that $A_k A_k^T = I - kk^T/|k|^2$. Thus it follows from the last equation that the projection method is equivalent to the Fourier method in which the pressure term has been eliminated. Similar arguments show that the projection method for two-dimensional inviscid flow reduces to Eq. (1.2).

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